Week 11
Generating Functions

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Section 1

Solving DP Problems quickly for Fun and Profit

How many ways are there to make change for N¢ with these coins?

(1), (5), (10), (25), (50)

For example, we have that 16¢ can be represented in 6 different ways:

$$\begin{aligned}
16¢ &= \textcircled{10}\ \textcircled{5}\ \textcircled{1} \\
&= \textcircled{5}\ \textcircled{5}\ \textcircled{5}\ \textcircled{1} \\
&= \textcircled{10}\ \textcircled{1}\ \textcircled{1}\ \textcircled{1}\ \textcircled{1}\ \textcircled{1}\ \textcircled{1} \\
&= \textcircled{5}\ \textcircled{5}\ \textcircled{1}\ \textcircled{1}\ \textcircled{1}\ \textcircled{1}\ \textcircled{1}\ \textcircled{1} \\
&= \textcircled{5}\ \underbrace{\textcircled{1} \cdots \textcircled{1}}_{11} \\
&= \underbrace{\textcircled{1} \cdots \textcircled{1}}_{16}
\end{aligned}$$

```python
@cache
def num_coin_sums(N, coins):
    if N < 0: return 0
    if N == 0: return 1
    if len(coins) == 0: return 0
    return num_coin_sums(N - coins[0], coins) +
    ↪  num_coin_sums(N, tuple(coins[1:]))
```

Consider a simpler problem - how many ways of making change with just dimes and nickels

To compact notation, let us denote $\textcircled{1}^4 = \textcircled{1}\textcircled{1}\textcircled{1}\textcircled{1}$

Consider all possible combinations of making change with $\textcircled{5}$ and $\textcircled{10}$

$$
\begin{aligned}
5¢ &: \textcircled{5} \\
10¢ &: \textcircled{10}, \textcircled{5}^2 \\
15¢ &: \textcircled{10}\,\textcircled{5}, \textcircled{5}^3 \\
20¢ &: \textcircled{10}^2, \textcircled{10}\,\textcircled{5}^2, \textcircled{5}^4 \\
&\vdots
\end{aligned}
$$

WARNING: Engineering levels of rigor ahead[1]

Consider the following sum of all combinations of nickels and dimes:

$$S = 1 + \textcircled{5}^{0} + \textcircled{5} + \textcircled{10} + \textcircled{5}^{2} + \textcircled{10}\,\textcircled{5} + \textcircled{5}^{3} + \textcircled{10}^{2} + \textcircled{10}\,\textcircled{5}^{2} + \textcircled{5}^{4} + \cdots$$

[1] We will resolve this later

$$
\begin{aligned}
S = 1 &+ \textcircled{5} + \textcircled{5}^2 + \textcircled{5}^3 + \textcircled{5}^4 + \textcircled{5}^5 + \cdots \\
&+ \textcircled{10} + \textcircled{10}\,\textcircled{5} + \textcircled{10}\,\textcircled{5}^2 + \textcircled{10}\,\textcircled{5}^3 + \textcircled{10}\,\textcircled{5}^4 + \textcircled{10}\,\textcircled{5}^5 + \cdots \\
&+ \textcircled{10}^2 + \textcircled{10}^2\,\textcircled{5} + \textcircled{10}^2\,\textcircled{5}^2 + \textcircled{10}^2\,\textcircled{5}^3 + \textcircled{10}^2\,\textcircled{5}^4 + \textcircled{10}^2\,\textcircled{5}^5 + \cdots \\
&+ \textcircled{10}^3 + \textcircled{10}^3\,\textcircled{5} + \textcircled{10}^3\,\textcircled{5}^2 + \textcircled{10}^3\,\textcircled{5}^3 + \textcircled{10}^3\,\textcircled{5}^4 + \textcircled{10}^3\,\textcircled{5}^5 + \cdots \\
&+ \textcircled{10}^4 + \textcircled{10}^4\,\textcircled{5} + \textcircled{10}^4\,\textcircled{5}^2 + \textcircled{10}^4\,\textcircled{5}^3 + \textcircled{10}^4\,\textcircled{5}^4 + \textcircled{10}^4\,\textcircled{5}^5 \cdots \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
S = 1 &\left(1 + \textcircled{5} + \textcircled{5}^2 + \textcircled{5}^3 + \textcircled{5}^4 + \textcircled{5}^5 + \cdots\right) \\
&+ \textcircled{10}\left(1 + \textcircled{5} + \textcircled{5}^2 + \textcircled{5}^3 + \textcircled{5}^4 + \textcircled{5}^5 + \cdots\right) \\
&+ \textcircled{10}^2\left(1 + \textcircled{5} + \textcircled{5}^2 + \textcircled{5}^3 + \textcircled{5}^4 + \textcircled{5}^5 + \cdots\right)\cdots \\
&+ \textcircled{10}^3\left(1 + \textcircled{5} + \textcircled{5}^2 + \textcircled{5}^3 + \textcircled{5}^4 + \textcircled{5}^5 + \cdots\right)\cdots \\
&+ \textcircled{10}^4\left(1 + \textcircled{5} + \textcircled{5}^2 + \textcircled{5}^3 + \textcircled{5}^4 + \textcircled{5}^5 + \cdots\right)\cdots \\
&\vdots
\end{aligned}
$$

$$S = \left(1 + \textcircled{5} + \textcircled{5}^2 + \textcircled{5}^3 + \cdots\right)\left(1 + \textcircled{10} + \textcircled{10}^2 + \textcircled{10}^3 + \cdots\right)$$

$$= \frac{1}{1 - \textcircled{5}} \cdot \frac{1}{1 - \textcircled{10}} \qquad \text{(geometric series)}$$

$$= \frac{1}{1 - x^5} \cdot \frac{1}{1 - x^{10}} \qquad \left(\textcircled{5} = x^5, \textcircled{10} = x^{10}\right)$$

Sanity Check

We can check that this power series actually works

```
sage: R.<x> = PowerSeriesRing(ZZ, default_prec=100)
sage: 1 / ((1-x^5) * (1-x^10))
1 + x^5 + 2*x^10 + 2*x^15 + 3*x^20 + 3*x^25 + 4*x^30 +
↪  4*x^35 + 5*x^40 + 5*x^45 + 6*x^50 + 6*x^55 + 7*x^60
↪  + 7*x^65 + 8*x^70 + 8*x^75 + 9*x^80 + 9*x^85 +
↪  10*x^90 + 10*x^95 + O(x^100)
```

For example, there are only 8 ways to make 70¢ with nickles and dimes
EXERCISE: Check this

By a similar kind of logic, letting C_n being the number of ways of making change for n¢, we have a **generating function**:

$$C(z) = \sum_{n \geq 0} C_n z^n = \frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})(1-z^{50})}$$

With some algebraic manipulation of this generating function[2], we can get an explicit formula for its coefficients

```
from math import comb as C

def num_coin_sums_fast(N):
    A = [1,2,4,6,9,13,18,24,31,39,45,52,57,63,67,69,69,67,63
    ↪ ,57,52,45,39,31,24,18,13,9,6,4,2,1]
    N //= 5
    q = N // 10; r = N % 10
    return A[r] * C(q+4, 4) + A[r+10] * C(q+3,4) + A[r+20] *
    ↪ C(q+2, 4) + A[r+30] * C(q+1, 4)
```

[2] See Knuth's *Concrete Mathematics* p345-6 where this example is taken for the details of the derivation - it's mostly (messy) algebra

Section 2

Ordinary Generating Functions

We now define things more rigorously - define a **combinatorial class** to be a set of objects with a corresponding **size function**

In our previous example, our combinatorial class was

$$\mathcal{C} = \left\{ \textcircled{50}^{z_1} \textcircled{25}^{z_2} \textcircled{10}^{z_3} \textcircled{5}^{z_4} \textcircled{1}^{z_5} : z_i \in \mathbb{Z}_{\geq 0} \right\}$$

and the associated size function is

$$\left\| \textcircled{50}^{z_1} \textcircled{25}^{z_2} \textcircled{10}^{z_3} \textcircled{5}^{z_4} \textcircled{1}^{z_5} \right\| = 50z_1 + 25z_2 + 10z_3 + 5z_4 + z_5$$

For any combinatorial class A, we define it's associated **ordinary generating function** to be

$$A(z) = \sum_{a \in A} z^{|a|} = \sum_{N \geq 0} A_N z^N$$

With this generating function, we can get the number of objects of size N by **extracting the corresponding coefficient**

$$A_N = [z^N]A(z)$$

In our last example, we had $C(z) = \frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})(1-z^{50})}$
We can use this to count the number of ways of getting 42¢ as follows:

$$\begin{aligned} C_{42} &= [z^{42}]C(z) \\ &= [z^{42}]\left(1 + z + z^2 + z^3 + z^4 + 2z^5 + \cdots + 31z^{41} + \underline{31z^{42}} + \cdots\right) \\ &= 31 \end{aligned}$$

Exercises

We will be working over binary strings - assume the size function is just the length of the string
Find the corresponding generating function for these combiatorial classes

- Σ = all binary strings
- $\mathcal{Z}$ = the set of all strings of zeros of length at least 5
- $\mathcal{E}$ = the set of all binary strings whose length is even

$$\Sigma(z) = \sum_{N \geq 0} 2^N z^N = \sum_{N \geq 0} (2z)^N = \frac{1}{1-2z}$$

$$\mathcal{Z}(z) = \sum_{N \geq 5} z^N = \frac{z^5}{1-z}$$

$$\mathcal{E}(z) = \sum_{N \geq 0} 2^{2N} z^{2N} = \frac{1}{1-4z^2}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes with associated generating functions $A(z)$ and $B(z)$ respectively.

We can combine these combinatorial classes to get new combinatorial classes with new generating functions

Let $\mathcal{C} = \mathcal{A} + \mathcal{B}$ be the disjoint union of $\mathcal{A}$ and $\mathcal{B}$. We have that the corresponding generating function is

$$C(z) = \sum_{c \in \mathcal{A}+\mathcal{B}} z^{|c|} = \sum_{a \in \mathcal{A}} z^{|a|} + \sum_{b \in \mathcal{B}} z^{|b|} = A(z) + B(z)$$

Similarly, we have that if $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ be the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$. then the corresponding generating function is

$$C(z) = \sum_{c \in \mathcal{A} \times \mathcal{B}} z^{|c|} = \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} z^{|a|+|b|} = \left(\sum_{a \in \mathcal{A}} z^{|a|} \right) \left(\sum_{b \in \mathcal{B}} z^{|b|} \right) = A(z)B(z)$$

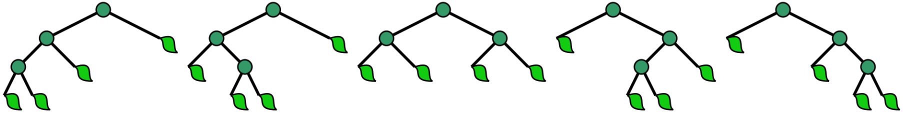
Finally, letting ϵ be the empty combinatorial class, let $\mathcal{C} = \epsilon + A + A^2 + A^3 + ... \equiv \text{SEQ}(A)$, we have that the corresponding generating function is $C(z) = \frac{1}{1-A(z)}$

With these new tools, we can solve the above problems much more easily.

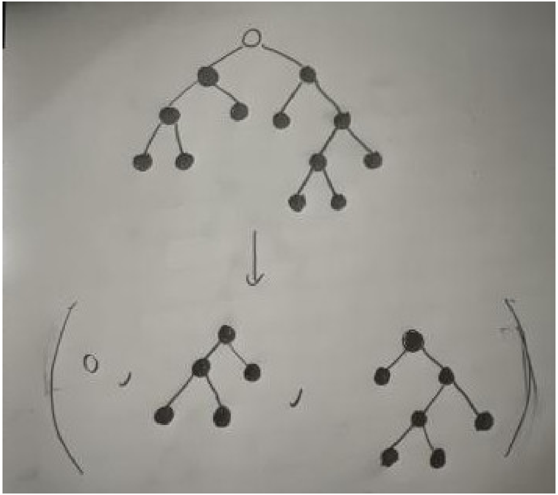
For example:

$$B = \{\texttt{0}, \texttt{1}\}, B(z) = 2z \implies \Sigma = \mathrm{SEQ}(B), \Sigma(z) = \frac{1}{1-2z}$$

Let's deal with something more complicated: how many binary trees are there with n internal nodes (and therefore $n + 1$ leaf nodes)?
An example for $n = 3$:



Let $\mathcal{T}$ be the combinatorial class of all such trees. Note that we can decompose any tree T as follows:



Note that we have an invertible mapping $T \mapsto (\circ, T_l, T_r)$ meaning that we can decompose and recompose tress uniquely.

Since the left and right subtrees are also in $\mathcal{T}$, we can use the above to come up with an equation defining $\mathcal{T}$ and get a generating function $T(z)$:

$$\mathcal{T} = \circ + \mathcal{T} \times \mathcal{T} \implies T(z) = z + T(z)^2 \implies T(z) = \frac{1 - \sqrt{1 - 4z}}{2}$$

With this generating function, we can get a general formula for the n^{th} coefficient T_n (Note that this is generally not possible)

$$(1-4z)^{1/2} = \sum_{k\geq 0} \binom{\frac{1}{2}}{k} (-4z)^k$$

$$\implies T(z) = \frac{1-\sqrt{1-4z}}{2} = -\frac{1}{2}\sum_{k\geq 1} \binom{\frac{1}{2}}{k} (-4z)^k$$

$$\implies T_n = -\frac{1}{2}\binom{\frac{1}{2}}{k}(-4)^k = \frac{1}{n}\binom{2n-2}{n-1}$$

These are the **Catalan numbers -** Richard Stanley has 207 examples of different sequences that correspond to the Catalan numbers

We conclude from an example from number theory - let p_n be the number of **partitions** of n, or the number of ways of writing n as a sum of positive integers.

For example, we have $p_5 = 7$ as

$$
\begin{aligned}
5 &= 5 \\
&= 4 + 1 \\
&= 3 + 2 \\
&= 3 + 1 + 1 \\
&= 2 + 2 + 1 \\
&= 2 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

Similar to our first example, we can define a generating function $P(z) = \sum_{k\geq 0} p_k z^k$

$$
\begin{aligned}
P(z) &= \sum_{k\geq 0} p_k z^k \\
&= (1 + z^1 + z^{1+1} + \cdots)(1 + z^2 + z^{2+2} + \cdots)(1 + z^3 + z^{3+3} + \cdots)\cdots \\
&= \prod_{j\geq 1} \frac{1}{1 - x^j}
\end{aligned}
$$

We end with a proof of a non-obvious fact

Let $P_o(n)$ be the number of partitions of n into *odd* parts. For example, we have that $P_o(7) = 5$ as

$$\begin{aligned} 7 &= 7 \\ &= 5 + 1 + 1 \\ &= 3 + 3 + 1 \\ &= 3 + 1 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1 + 1 + 1 \end{aligned}$$

Next, let $P_d(n)$ be the number of partitions of n into *distinct* parts. For example, we have that $P_d(7) = 5$ as

$$
\begin{aligned}
7 &= 7 \\
&= 6 + 1 \\
&= 5 + 2 \\
&= 4 + 3 \\
&= 4 + 2 + 1
\end{aligned}
$$

This equality is not a coincidence - we will show that $P_o(n) = P_d(n)$ for any n.
Let

$$P_o(z) = \sum_{k\geq 0} P_o(k) z^k \quad , \quad P_d(z) = \sum_{k\geq 0} P_d(k) z^k$$

be the corresponding generating functions.

The Proof

We have:

$$
\begin{aligned}
P_d(z) &= (1+z)(1+z^2)(1+z^3)(1+z^4)(1+z^5)\cdots \\
&= \frac{1-z^2}{1-z}\cdot\frac{1-z^4}{1-z^2}\cdot\frac{1-z^6}{1-z^3}\cdot\frac{1-z^8}{1-z^4}\cdot\frac{1-z^{10}}{1-z^5}\cdots \\
&= \frac{1}{(1-z)(1-z^3)(1-z^5)\cdots} \\
&= (1+z^1+z^{1+1}+z^{1+1+1}+\cdots)(1+z^3+z^{3+3}+z^{3+3+3}+\cdots) \\
&\quad (1+z^5+z^{5+5}+z^{5+5+5}+\cdots)\cdots \\
&= P_o(z)
\end{aligned}
$$

Since these two sequences have the same generating function, their coefficients must be the same - ending the proof.

Further Resources

- *generatingfunctionology* by Herbert Wilf - good overall resource on generating functions
- *Analytic Combinatorics* by Sedgewick and Flajolet - longer resource on generating functions that details the symbolic method (detailed in the presentation) and how to deal with generating functions using complex analysis to get asymptotic information
- *Concrete Mathematics* by Graham, Knuth and Patashnik - the Bible on any mathematics you may need for computer science; has a chapter on generating functions that was referenced

A generating function is a clothesline on which we hang up a sequence of numbers for display.

— HERBERT WILF (1990)