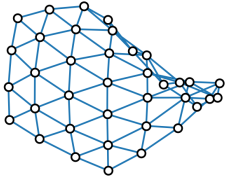
Graph Drawing and Tutte's Theorem

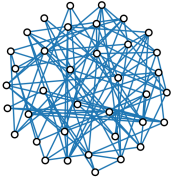
Ian Chen



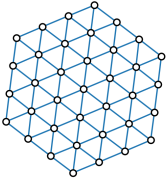
Graphs



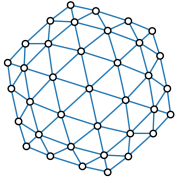
Spring



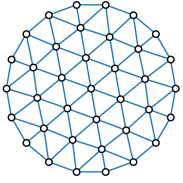
ARF



Kamada–Kawai



Spectral



Tutte



Outline

Overview
- Tutte
- Other force-directed

Tutte's Theorem



Overview

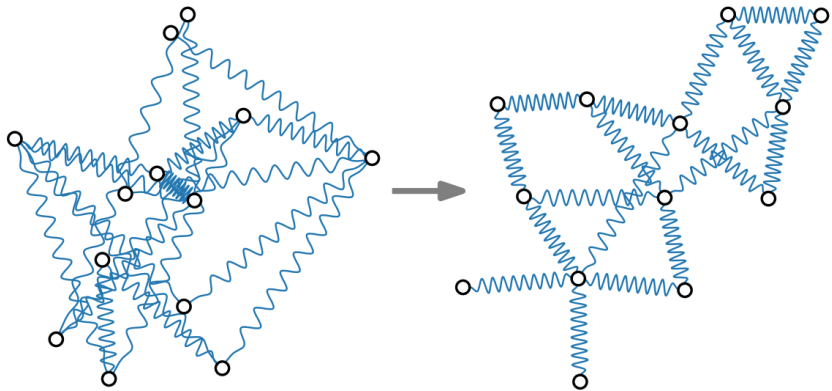
- Tutte
- Other force-directed

Tutte's Theorem



Tutte's Spring Embedding

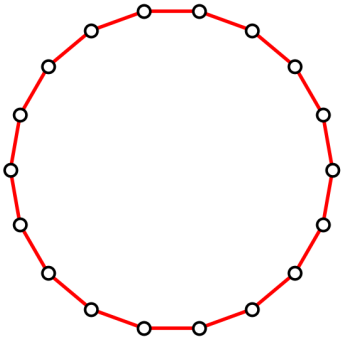
- Graph $G = (V, E)$, embedding $p : V \to \mathcal{R}^2$ (denote $\boldsymbol{p}_v = p(v)$).

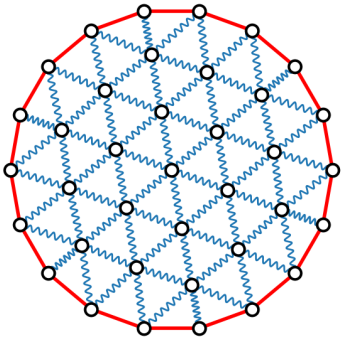




Tutte's Spring Embedding (cont.)

- boundary S positioned at convex polygon; free vertices $V \setminus S$







Tutte's Spring Embedding (cont.)

- Equilibrium condition

$$\sum_{v \in N(u)} k_{uv}(\boldsymbol{p}_v - \boldsymbol{p}_u) = \mathbf{0} \qquad \forall u \notin S$$

- Energy-minimization condition

$$\min \sum_{u \notin S} \sum_{v \in N(u)} \frac{1}{2} k_{uv} \|\boldsymbol{p}_v - \boldsymbol{p}_u\|_2^2$$

- Newton-Raphson / Gaussian-Elimination

$$\boldsymbol{F}_u^t = -\deg(u)\boldsymbol{p}_u^t + \sum_{v \in N(u)} k_{uv}\boldsymbol{p}_v^t$$

$$\boldsymbol{p}_u^{t+1} = \boldsymbol{p}_u^t + \frac{1}{\deg(u)}\boldsymbol{F}_u^t$$



Tutte's Spring Embedding (cont.)

- Theorem: (Whitney '33) every 3-connected planar graph has a unique embedding[1]
- Theorem: (Tutte '63) Tutte embedding is strongly-convex embedding (if G 3-connected).



[1] up to homeomorphism, on the sphere

Kamada-Kawai layout

- spring equilibrium based on graph distances d_{uv}
- spring stiffness k, scale L

$$f_{uv} = -\frac{k}{d_{uv}^2}(\|\boldsymbol{p}_v - \boldsymbol{p}_u\| - Ld_{uv})$$

$$\boldsymbol{F}_u = \sum_{v \in N(u)} f_{uv} \frac{\boldsymbol{p}_v - \boldsymbol{p}_u}{\|\boldsymbol{p}_v - \boldsymbol{p}_u\|}$$

$$E = \frac{1}{2} \sum_{uv \in E} \frac{k}{d_{uv}^2}(\|\boldsymbol{p}_v - \boldsymbol{p}_u\| - Ld_{uv})^2$$



Anti-gravity drawing

- scale L, anti-gravity around vertices (repulsion)

$$f_{uv}^{\text{attr}} = \frac{\|\boldsymbol{p}_v - \boldsymbol{p}_u\|^2}{L} \qquad f_{uv}^{\text{rep}} = \frac{L^2}{\|\boldsymbol{p}_v - \boldsymbol{p}_u\|}$$

$$\boldsymbol{F}_u = \sum_{v \in N(u)} f_{uv}^{\text{attr}} \frac{\boldsymbol{p}_v - \boldsymbol{p}_u}{\|\boldsymbol{p}_v - \boldsymbol{p}_u\|} + \sum_{v \in V} f_{uv}^{\text{rep}} \frac{\boldsymbol{p}_v - \boldsymbol{p}_u}{\|\boldsymbol{p}_v - \boldsymbol{p}_u\|}$$



Overview
Tutte
Other force-directed

Tutte's Theorem



Tutte's Spring Embedding (cont.)

- Theorem: (Whitney '33) every 3-connected planar graph has a unique embedding[2]
- Theorem: (Tutte '63) Tutte embedding is strongly-convex embedding (if G 3-connected).



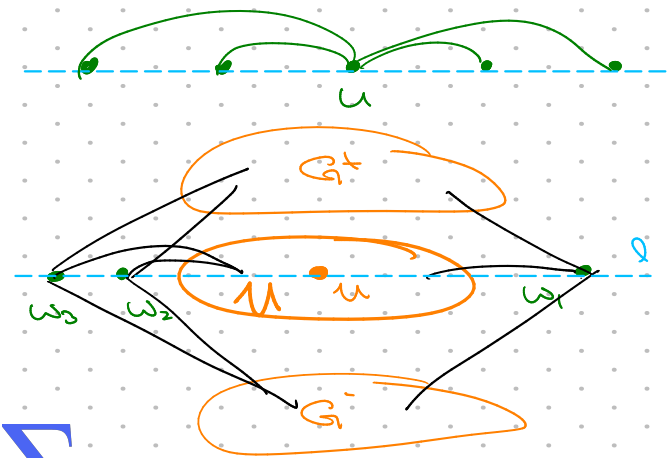
[2] up to homeomorphism, on the sphere

Proof outline

- No degenerate neighborhoods (not all neighborhood lies on a line)
  - $\Longrightarrow$ every free vertex has neighbors strictly on both sides of any line
  - $\Longrightarrow$ all free vertices inside the outer polygon
- Split-faces lemma (adjacent faces strictly separated by a line)
  - $\Longrightarrow$ strictly-convex faces
  - $\Longrightarrow$ no degenerate vertices (embedding injective)
- Points are in exactly one face
  - $\Longrightarrow$ planar embedding



No degenerate Neighborhoods



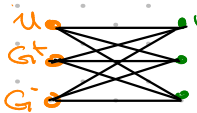
Equilibrium condition:

$$\sum_v (p_v - p_u) = 0$$

Obs: every vertex has nondecreasing path to T (by height)

$\Rightarrow G^+, G^-$ connected

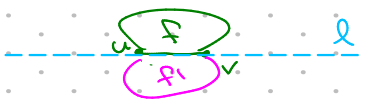
$U = \{ v \in V : N(v) \text{ on } \ell \}$



$K_{3,3}$ minor $\Rightarrow$ not planar ⚠

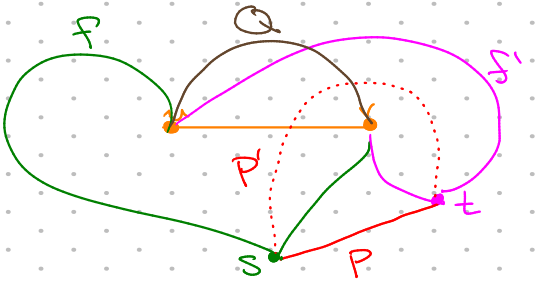
$\sum$ assuming $N(u)$ is degenerate

Split-Faces Lemma:



f and f' are on (strictly) separate sides of l

Equilibrium Condition:
$\sum_v (P_v - P_u) = 0$



① P' curve in $\mathbb{R}^2$ from s-t contained in f, f'

② Any Q (u→v) must intersect P (s→t)

③ If s, t on same side:
⇒ choose P in V^-
⇒ choose Q in V^+

∎

$\sum$ position vertices to any point displacing