The PACE 2023 Challenge: Twin-Width

Anakin

Outline

What is Twin-width?

Computing Twin-width

PACE 2023

Section 1

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- Algorithmic Complexity
- Efficient encodings
- *Decomposition*

Operations on Graphs: Disjoint Union

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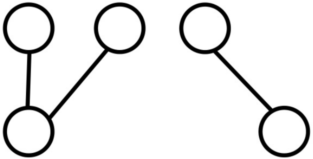
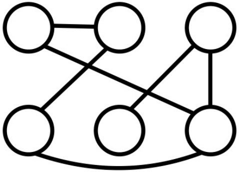


Operations on Graphs: Complement



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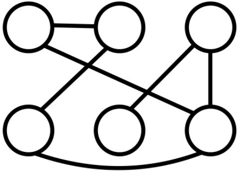

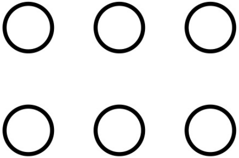

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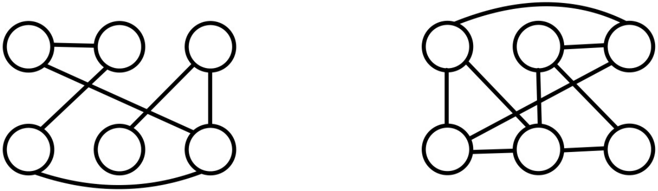

Cographs

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- K_1 is a cograph
- The disjoint union of two cographs is also a cograph
- The complement of a cograph is a cograph

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This is called *twin-width* [BKTW20]

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- Our edges in E will have a color associated with them: black or red.
  - The red edges will be our error that we want
- Vertices are black neighbors if linked by a black edge, and red neighbors if linked by a red edge

Contractions by picture

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All other edges are left alone and maintain their color

Twin-width

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The twin-width of G is the minimum d such that there exists a d-sequence of G

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The number of red edges may increase or decrease

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If we have the d-sequence for a graph, we can decompose the graph into complete bipartite graphs and do breadth- rst-search in $O(n \log n)$ time [BGK+20]

This works even if the number of edges is $\Omega(n^2)$

Section 2

Computing Twin-width

It's Hard

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There are some things known for special cases

- Cographs are the graphs with twin-width zero
- *d*-dimensional graphs have twin-width $3d$ [BKTW20]
- Planar graphs have twin-width 9, and bipartite planar graphs have twin-width 6 [Hli22]
- Graphs with 4 vertices have twin-width 1 and graphs with 5 vertices have twin-width 2 [Das22]
- In general, bipartite graphs may have arbitrarily large twin-width

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[SS21] found a way to encode twin-width into a SAT formula

This is one of the only ways we can reasonable compute twin-widths

Section 3

PACE 2023

A Programming Competition

Parameterized **A**lgorithms and **C**omputational **E**xperiments is a long-term programming competition

Our goal will be to devise an efficient algorithm to compute the twin-width d of arbitrary graphs and their d-sequence

- Exact Track: Compute the exact sequence
- Heuristic Track: Compute an approximate sequence

The Club Submission

I only found out about this recently so we are a bit behind

We should make on submission on one track

- I was thinking about focusing on the exact track

There are 100 public test cases + 100 test cases

Score = the number of test cases we can solve in a given time limit

My goal is just to put together something and see how well we do

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